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SUNDUAL CHARACTERIZATIONS OF THE TRANSLATION GROUP OF $\mathbb{R}$

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ABSTRACT. We characterize the first three sundual spaces of $C_0(\mathbb{R})$, with respect to the translation group of $\mathbb{R}$.

1. INTRODUCTION

Let $G = \{T(t) : t \geq 0\}$ be a (*strongly continuous*) C_0-semigroup of bounded, linear operators which act on a real or complex Banach space E. The dual semigroup, $G^* = \{T^*(t) : t \geq 0\}$, defines a semigroup of bounded operators on the Banach dual E^* of E.

If E is not reflexive, then G^* need not be a strongly continuous semigroup, although G^* is always a $\sigma(E^*, E)$-continuous semigroup. For example, let $E = C_0(\mathbb{R})$ be the space of continuous functions on $\mathbb{R}$ that vanish at infinity. Suppose the action of G on $C_0(\mathbb{R})$ is given by translation, $(\forall x, t \in \mathbb{R})$ $(\forall f \in C_0(\mathbb{R}))$,

$$T(t)f(x) = f(x + t).$$

If δ_x is the point measure at $x \in \mathbb{R}$, then clearly $\delta_x \in C_0^*(\mathbb{R})$, where $C_0^*(\mathbb{R})$ is the Banach dual of $C_0(\mathbb{R})$. The action of the dual family G^* on δ_x is given by $T^*(t)\delta_x = \delta_{x+t}$. It follows that $\|T^*(t)\delta_x - \delta_x\| = 2$, when $t \neq 0$.

The example above led R.S. Phillips ([Phi]) to define the following linear subspace:

$$E^{\odot} = (E, G)^{\odot} = \{\phi \in E^* : \lim_{t \to 0} \|T^*(t)(\phi) - \phi\| = 0\}.$$

$E^{\odot}$ is called the *sundual* of E with respect to G.

Phillips showed that $(E, G)^{\odot}$ is the largest closed, G^*-invariant linear subspace of E^*, on which the adjoint family G^* acts as a strongly continuous semigroup. In addition, $(E, G)^{\odot}$ is a norm-fundamental linear subspace, which implies that $(E, G)^{\odot}$ is weak*-dense in E^*.

Again, consider the space $C_0(\mathbb{R})$ and the group G defined on $C_0(\mathbb{R})$ by *translation*. Recall that $C_0^*(\mathbb{R})$ is identified with the space $M(\mathbb{R})$ of all bounded, regular, complex, Borel measures on $\mathbb{R}$. Let $\phi \in C_0^{\odot}(\mathbb{R})$ and let μ be the unique measure that represents ϕ. We can conclude, using a result of A. Plessner ([Ple]) that characterized absolutely continuous functions, that $\phi \in C_0^{\odot}(\mathbb{R})$ if and only if μ is absolutely

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continuous with respect to Lebesgue measure $[dx]$ of $\mathbb{R}$. Hence, $(C_0(\mathbb{R}), G)^{\odot}$ can be identified with the space $L^1(\mathbb{R}, dx)$.

Observe that the *sundual* of $L^1(\mathbb{R}, dx)$ is a closed, linear subspace of the dual space $L^\infty(\mathbb{R}, dx)$ of $L^1(\mathbb{R}, dx)$. Therefore, the *sundual* of $L^1(\mathbb{R}, dx)$ can be identified with $C_0^{\odot\odot}(\mathbb{R})$, the second *sundual* of $C_0^{\odot}(\mathbb{R})$.

Let $BUC_c(\mathbb{R})$ denote the space of all bounded, uniformly continuous, complex-valued functions on $\mathbb{R}$, with sup-norm. It is clear that $BUC_c(\mathbb{R})$, by definition, is a subspace of the second *sundual* of $(C_0(\mathbb{R}), G)$. We claim that the second *sundual* of $(C_0(\mathbb{R}), G)$ can be *identified* with $BUC_c(\mathbb{R})$. To see this, let $f \in C_0^{\odot\odot}(\mathbb{R}) \subseteq L^\infty(\mathbb{R})$, and for all $a \neq 0$, define

$$F_a(x) = \frac{1}{a}\int_0^a T(t)(f)(x)\ dt,\ \ x \in \mathbb{R}.$$

Then F_a is a bounded function that satisfies a *Lipschitz condition*, with constant $2\frac{\|f\|_\infty}{|a|}$; thus, $\forall a \neq 0$, $F_a \in (BUC(\mathbb{R}))_c$. Moreover,

$$\|F_a - f\|_\infty = \left\|\frac{1}{a}\int_0^a (T(t)f - f)\ dt\right\|_\infty \leq \frac{1}{|a|}\left|\int_0^a \|T(t)f - f\|_\infty\ dt\right|$$

shows that $\lim_{a\to 0}\|F_a - f\|_\infty = 0$. Therefore, the family of functions F_a, with $a \neq 0$, converges uniformly to a function $F \in (BUC(\mathbb{R}))_c$. Moreover, F satisfies $\|F - f\|_\infty = 0$, i.e., $F \equiv f$ a.e. $[dx]$.

The identification of $(C_0^{\odot\odot}(\mathbb{R}), G)$ with $BUC_c(\mathbb{R})$ makes it possible to investigate a characterization of the *sundual* of $BUC_c(\mathbb{R})$ with respect to the adjoint group G^*. The primary goal of this paper is to accomplish this very purpose. We devote four sections to our investigation. In Section 2 we obtain several descriptions of $BUC_c^*(\mathbb{R})$; in Section 3 we introduce a special subspace of $BUC_c^*(\mathbb{R})$. This space consists of *translation-invariant, linear functionals*; we call the space FIX. We end Section 3 by providing examples of elements in FIX. In Section 4 we introduce a second subspace of $BUC_c^*(\mathbb{R})$; we denote this space $SFIX$. $SFIX$, like FIX, plays an important role in helping us to understand the structure of $BUC_c(\mathbb{R})^{\odot}$. Finally, in Section 5 we provide a characterization of $BUC_c(\mathbb{R})^{\odot}$.

2. Some properties of $BUC_c^*(\mathbb{R})$ and $BUC_c(\mathbb{R})^{\odot}$

Let G be a semigroup (group) of operators defined on a Banach space E and let G^* be its dual semigroup (group). In order to characterize the *sundual* of E relative to G, one must determine the relationship between the structure of E^* and the action of G^* on E^*.

Let $E = BUC_c(\mathbb{R})$ and let G be the translation group defined on E. In this section, we will provide some properties of $BUC_c^*(\mathbb{R})$ that will allow us to derive basic properties of $BUC_c(\mathbb{R})^{\odot}$. But first, we will establish the fact that there is no loss of generality in our results if we consider the space $BUC(\mathbb{R})$ of bounded, *real-valued*, uniformly continuous functions defined on $\mathbb{R}$.

One can easily verify that under the pointwise operations of addition, scalar multiplication, and multiplication, $BUC(\mathbb{R})$ equipped with sup-norm is a Banach algebra with unit e. $BUC(\mathbb{R})$ is also a Banach lattice. Furthermore, it is a special example of an **M**-space, in the sense of Kakutani, with *strong order unit* e (see [Ka1]). Recall that the Banach dual of an **M**-space is a Dedekind complete **L**-space. Thus, the Banach dual of $BUC(\mathbb{R})$ is equipped with a *dual norm* that is *order continuous*, additive on the *cone* of positive, linear functionals, and is a function of the modulus of the dual space.

In the case of $BUC_c(\mathbb{R})$, the modulus of an element $\phi \in BUC_c^*(\mathbb{R})$ is the positive, linear functional which satisfies, $\forall\ 0 \le f \in BUC_c(\mathbb{R})$,

$$|\phi|(f) = \sup\left(|\phi(g)| : g \in BUC_c(\mathbb{R}) \text{ and } |g| \le f\right).$$

The norm satisfies

$$\|\phi\| = \sup\left(|\phi(g)| : g \in BUC_c(\mathbb{R}) \text{ and } \|g\| \le 1\right)$$

$$= \sup\left(|\phi|(g) : g \in BUC_c(\mathbb{R}) \text{ and } |g| \le e\right) = |\phi|(e).$$

The norm is called the *total variation* of ϕ; clearly, it is additive on the cone of positive, linear functionals.

Observe that $BUC_c(\mathbb{R}) = BUC(\mathbb{R}) + iBUC(\mathbb{R})$. If $z = f + ig \in BUC_c(\mathbb{R})$, with $f, g \in BUC(\mathbb{R})$, then its modulus $|z|$ is given by, $\forall x \in \mathbb{R}$,

$$|z|(x) = |z(x)| = \sqrt{f(x)^2 + g(x)^2}$$

$$= \sup\left(|f(x)\cos\theta + g(x)\sin\theta| : 0 \le \theta < 2\pi\right),$$

and its norm is given by $\|z\|_c = \|z\|_\infty$. In other words, in the sense of Banach lattice theory ([Zan]), $BUC_c(\mathbb{R})$ is the complexification of $BUC(\mathbb{R})$.

From the above discussion, we see that $BUC_c^*(\mathbb{R})$ is the *complexification space* of $BUC^*(\mathbb{R})$. That is, $BUC_c^*(\mathbb{R}) = BUC^*(\mathbb{R}) + iBUC^*(\mathbb{R})$. Hence, for any $\phi_1 + i\phi_2 = \phi \in BUC_c^*(\mathbb{R})$, with $\phi_1, \phi_2 \in BUC^*(\mathbb{R})$, the modulus $|\phi|$ of ϕ is the positive, linear functional that satisfies

$$|\phi| = \sup\left(|\phi_1\cos\theta + \phi_2\sin\theta| : 0 \le \theta < 2\pi\right), \tag{1}$$

$$\forall\ 0 \le f \in BUC(\mathbb{R}),\ |\phi|(f) = \sup\left(|\phi(g)| :\ g \in BUC_c(\mathbb{R}) \text{ and } |g| \le f\right), \tag{2}$$

$$\sup(|\phi|_1, |\phi|_2) \le |\phi| \le |\phi|_1 + |\phi|_2 \text{ and } \big||\phi|_1 - |\phi|_2\big| \le |\phi|. \tag{3}$$

The dual norm of $BUC_c^*(\mathbb{R})$ relates to the dual norm of $BUC^*(\mathbb{R})$ in the following way:

$$\|\phi\|_c = \sup\left(|\phi|(f) : f \in BUC(\mathbb{R}) \text{ and } \|f\| \le 1\right) \tag{4}$$

$$= \sup\left(|\phi(f)| : f \in (BUC(\mathbb{R}))_c \text{ and } |f| \le e\right) = |\phi|(e) = \|\ |\phi|\ \|.$$

It follows from the definition of the translation group G that the operators in G are linear isometries that leave $BUC(\mathbb{R})$, the real part of $BUC_c(\mathbb{R})$, invariant. Furthermore, the operators in G are lattice isomorphisms and are multiplicative on both $BUC_c(\mathbb{R})$ and $BUC(\mathbb{R})$.

For the adjoint group G^* of G, we have:

Theorem 2.1. *The operators in G^* are linear isometries that leave $BUC(\mathbb{R})$, the real part of $BUC_c(\mathbb{R})$, invariant. The restriction of G^* to $BUC^*(\mathbb{R})$ acts as a group of linear lattice isomorphisms. Furthermore,* $(\forall \phi \in BUC_c^*(\mathbb{R}))$ $(\forall t \in \mathbb{R})$, $|T^*(t)(\phi)| = T^*(t)(|\phi|)$.

Proof. The first part of the theorem follows from the fact that the dual of an isometry is an isometry. The second part follows from the fact that an invertible operator defined on a Banach lattice is a lattice isomorphism if and only if its inverse is a positive linear operator. To illustrate the last part of the theorem, observe that $(\forall t \in \mathbb{R})$ $(\forall 0 \leq f \in BUC^*(\mathbb{R}))$, we have

$$|T^*(t)(\phi)|(f) = \sup\left(|T^*(t)(\phi)(g)| : g \in BUC_c(\mathbb{R}) \text{ and } |g| \leq f\right)$$

$$= \sup\left(|\phi(T(t)g)| : g \in BUC_c(\mathbb{R}) \text{ and } |g| \leq f\right) = |\phi|(T(t)f) = T^*(t)|\phi|(f).$$

Hence, $|T^*(t)(\phi)| = T^*(t)(|\phi|)$ and $\|T^*(t)(\phi)\|_c = |\phi(e)|$. □

We now show that there is no loss of generality if we consider the space $BUC(\mathbb{R})$ instead of $BUC_c(\mathbb{R})$.

Theorem 2.2. *If $\phi = \phi_1 + i\phi_2$, with $\phi_1, \phi_2 \in BUC^*(\mathbb{R})$, then $\phi \in (BUC^{\odot}(\mathbb{R}))_c$ if and only if $\phi_1, \phi_2 \in BUC^{\odot}(\mathbb{R})$. Furthermore, if $\phi \in BUC_c(\mathbb{R})^{\odot}$, then $|\phi| \in BUC^{\odot}(\mathbb{R})$, and $BUC_c(\mathbb{R})^{\odot}$ is the complexification space of $BUC^{\odot}(\mathbb{R})$.*

Proof. The first part of the theorem follows from (1)–(4). Indeed, if $\phi = \phi_1 + i\phi_2$, with $\phi_1, \phi_2 \in BUC^*(\mathbb{R})$, and $t \in \mathbb{R}$, then

$$\max\left(\|T^*(t)\phi_1 - \phi_1\|, \|T^*(t)\phi_2 - \phi_2\|\right) \leq \|T^*(t)\phi - \phi\|$$

$$\leq \|T^*(t)\phi_1 - \phi_1\| + \|T^*(t)\phi_2 - \phi_2\|.$$

If $\phi \in BUC_c(\mathbb{R})^{\odot}$, then by (3) and Theorem 2.1, we have

$$|T^*(t)|\phi| - |\phi|| = ||T^*(t)\phi| - |\phi|| \leq |T^*(t)\phi - \phi|,$$

and so $\phi \in BUC^{\odot}(\mathbb{R})$. In other words, $BUC_c(\mathbb{R})^{\odot}$ is the complexification space of $BUC^{\odot}(\mathbb{R})$. □

We will use the following definitions throughout the remainder of this paper.

Definition 2.3.
- U is an **order ideal** in X if $x \in X$ and $u \in U$ with $|x| \leq |u|$, implies $x \in U$.
- An order ideal B is a **band** in X if and only if every positive, upward directed system $(u_\tau) \subseteq B$ has the property that if $u = \sup_\tau(u_\tau)$ exists in X, then $u \in B$.
- X has **order continuous norm** ρ if $0 \leq x_\alpha \uparrow x$ implies $\rho(x_\alpha - x) \to 0$ ([Zan]).

It follows from Definition 2.3 that every Banach lattice with *order continuous norm* is *Dedekind complete*. We also have,

Theorem 2.4. *If S is a Dedekind-complete, vector lattice, then the following hold:*
- *If A_1 and A_2 are bands in S and $A_1 \perp A_2$ (i.e., $(\forall x \in A_1)$ $(\forall y \in A_2)$, $\inf(|x|, |y|) = 0$), then $A_1 \oplus A_2$ is a band.*
- *Every band B in S is a projection band, i.e., $S = B \oplus B^d$, where*

$$B^d = \{y \in S : \forall x \in B,\ \inf(|y|, |x|) = 0\}$$

([LuZ]*).*

It is a natural question to ask whether the *sundual* of a Banach lattice, with respect to a C_0-group (or semigroup) of positive, linear operators, inherits any of the *order properties* of its Banach dual. In general, this is not the case. But in the case of $BUC(\mathbb{R})$, it is clear that $BUC^{\odot}(\mathbb{R})$ is a linear, *vector sublattice* of $BUC^*(\mathbb{R})$ and

$$(BUC^{\odot}(\mathbb{R}))_c = BUC_c(\mathbb{R})^{\odot}.$$

We will need Definition 2.3 and the following theorem of B. de Pagter to prove Theorem 2.5: Suppose E is a Banach lattice with the property that its Banach dual, E^*, has order continuous norm. If $H = \{U(t) : t \geq 0\}$ is a C_0-semigroup of positive, linear operators on E, then the sundual $E^{\odot}$ of E, with respect to H, is a **band** in E^* ([Pag]).

Theorem 2.5. *With respect to the translation group G, $BUC^{\odot}(\mathbb{R})$ is a* **projection band** *of $BUC^*(\mathbb{R})$, i.e.,*

$$BUC^*(\mathbb{R}) = BUC^{\odot}(\mathbb{R}) \oplus \left(BUC^{\odot}(\mathbb{R})\right)^d.$$

Furthermore, $BUC^{\odot}(\mathbb{R})$ and $(BUC^{\odot}(\mathbb{R}))^d$ are **invariant** *with respect to the adjoint group G^*.*

Proof. Since $BUC^*(\mathbb{R})$ is an abstract **L**-space, then $BUC^*(\mathbb{R})$ has *order continuous norm*. Thus, the first part of the theorem follows from the result of de Pagter.

The fact that $(BUC^{\odot}(\mathbb{R}))^d$ is G^*-invariant follows from the fact that the operators $T(t) \in G^*$, $t \in \mathbb{R}$, are lattice preserving. □

We obtain yet another decomposition of $BUC^*(\mathbb{R})$ by utilizing the polar set

$$C_0^{\perp}(\mathbb{R}) = \{\phi \in BUC^*(\mathbb{R}) : \phi(C_0^{\odot}(\mathbb{R})) = 0\}.$$

The space $C_0(\mathbb{R})$ is a norm-closed ideal in $BUC(\mathbb{R})$; thus, its polar set is a projection band in $BUC^*(\mathbb{R})$. Therefore, $BUC^*(\mathbb{R}) = C_0^{\perp}(\mathbb{R}) \oplus C_0^{\perp}(\mathbb{R})^d$. Note that the dual space $C_0^*(\mathbb{R})$ can be expressed as $C_0^*(\mathbb{R}) = BUC^*(\mathbb{R})/C_0^{\perp}(\mathbb{R})$ ([Ru]). It follows that $(C_0^{\perp}(\mathbb{R}))^d \cong C_0^*(\mathbb{R})$, so that $(C_0^{\perp}(\mathbb{R}))^d$ can be *identified* with the **L**-space $M(\mathbb{R})$ of all bounded, regular, Borel measures on $\mathbb{R}$, with total variation norm.

The band $C_0^{\perp}(\mathbb{R})$ represents the dual of the quotient space $BUC(\mathbb{R})/C_0(\mathbb{R})$; thus, it can be represented by an **L**-space consisting of *purely, finitely additive* measures. These measures are defined on the σ-algebra of Borel subsets of $\mathbb{R}$. The measures are *finitely additive*, because they vanish on the compact subsets of $\mathbb{R}$.

From the Radon-Nikodym theorem, it follows that $L^1(\mathbb{R})$ is a band of $M(\mathbb{R})$; hence, it is a projection band of $BUC^*(\mathbb{R})$. Also, since $L^1(\mathbb{R})$ is identified with $C_0^{\odot}(\mathbb{R})$ ([Ple]), then $L^1(\mathbb{R})$ is a projection band of $BUC^{\odot}(\mathbb{R})$. The disjoint complement of $C_0^{\odot}(\mathbb{R})$ in $M(\mathbb{R})$ consists of the *singular* measures. Let $M_s(\mathbb{R})$ denote the *singular* measures. Then $M_s(\mathbb{R})$ is a band. In summary,

Theorem 2.6.

$$BUC^*(\mathbb{R}) = C_0^{\perp}(\mathbb{R}) \oplus C_0^{\perp}(\mathbb{R})^d$$

$$= L^1(\mathbb{R}) \oplus M_s(\mathbb{R}) \oplus C_0^{\perp}(\mathbb{R}).$$

Furthermore, $L^1(\mathbb{R}) \subseteq BUC^{\odot}(\mathbb{R})$.

For the remainder of this paper, we shall focus on the disjoint complement of $L^1(\mathbb{R})$ in $BUC^{\odot}(\mathbb{R})$. To this end, we shall need another representation of $BUC^*(\mathbb{R})$.

From the Gelfand representation theorem and Kakutani ([Ka1]), it follows that there exists a compact Hausdorff space Ω, such that $BUC(\mathbb{R})$ is *order/algebraic* isomorphic and isometric with the space $C(\Omega)$ of all bounded, continuous, real-valued functions on Ω. The space Ω can be identified in the following way: let

$$F = \{\phi \in BUC^*(\mathbb{R}) : 0 \leq \phi \text{ and } \|\phi\| = \phi(e) = 1\}.$$

Observe, that F is a convex, weak*-compact subset of the unit ball of $BUC^*(\mathbb{R})$. Let $Ext(F)$ be the set of extreme points of F. Then Ω can be identified with the set $Ext(F)$.

Recall that a positive, linear functional ϕ is called an *atom* whenever $0 \leq \psi \leq \phi$ implies that $\psi = a\phi$, for some $0 \leq a \leq 1$. We have the following equivalences: $\forall \phi \in F$,

1. $\phi \in Ext(F) = \Omega$.
2. ϕ is *multiplicative*, i.e., $\forall f, g \in BUC(\mathbb{R})$, $\phi(f)\phi(g) = \phi(fg)$.
3. ϕ is a *linear, lattice homomorphism*, i.e., $\forall f \in BUC(\mathbb{R})$, $|\phi(f)| = \phi(|f|)$.
4. ϕ is an *atom* and $\phi(e) = 1$.

Note that if $x \in \mathbb{R}$, then the one-point measure δ_x, defined as $\forall f \in BUC(\mathbb{R})$, $\delta_x(f) = f(x)$, is an *atom*. The mapping $x \to \delta_x$ of $\mathbb{R}$ into F is in fact an imbedding of $\mathbb{R}$ into Ω. Furthermore, the mapping $f \to \hat{f}$ of $BUC(\mathbb{R})$ onto $C(\Omega)$, where $\hat{f}$ is the unique extension of f over Ω, is the Gelfand representation mapping. If we consider the *uniformity* on $\mathbb{R}$ defined by the family of semimetrics,

$$\{|f(x) - f(y)|, x, y \in \mathbb{R} : f \in BUC(\mathbb{R})\},$$

then Ω is actually the *completion* of $\mathbb{R}$, with respect to this *uniformity.*

Let $M(\Omega)$ be the space of all bounded, regular, Borel measures on Ω. From the above discussion, we see that $M(\Omega) \cong C^*(\Omega)$. Thus, if $\omega \in \Omega$, then we can regard δ_ω as an *atom* of $M(\Omega)$. We can also regard δ_ω as a positive, linear functional in $C^*(\Omega)$, i.e., $\forall f \in BUC(\mathbb{R})$,

$$\delta_\omega(f) = \hat{f}(\omega).$$

Recall that a measure can be written as the disjoint sum of its *discrete* and *diffuse* parts. We say that an *order-bounded*, linear functional is *discrete* if it is the *band* spanned by the atomic, linear functionals; it is *diffuse* if it is nonatomic, i.e., if it is disjoint from all the atoms. Hence,

Theorem 2.7. *The band in $BUC^*(\mathbb{R})$ that is spanned by its atoms is linear, lattice-isomorphic, and isometric with $l^1(\Omega)$. The disjoint complement $(l^1(\Omega))^d$, denoted by $DF(BUC^*(\mathbb{R}))$, is the band of the nonatomic elements of $BUC^*(\mathbb{R})$. We have*

$$BUC^*(\mathbb{R}) = l^1(\Omega) \oplus DF(BUC^*(\mathbb{R})).$$

Furthermore, $l^1(\Omega)$ and $DF(BUC^*(\mathbb{R}))$ are G^*-invariant.

Proof. Observe that if $\phi \in l^1(\Omega)$, then

$$\phi = \sum_{n=1}^{\infty} \lambda_n \delta_{\omega n},$$

with $\sum_{n=1}^{\infty} |\lambda_n| < \infty$ and $\forall n$, $\omega_n \in \Omega$. Since the family of atoms are mutually disjoint, it follows that

$$|\phi| = \sum_{n=1}^{\infty} |\lambda_n| \delta_{\omega n};$$

thus,

$$\|\phi\| = |\phi|(e) = \sum_{n=1}^{\infty} |\lambda_n|$$

is equal to the norm of ϕ as an element of $l^1(\Omega)$.

If ϕ is an atom and $0 \le \psi \le T^*(t)(\phi)$, then $0 \le T^*(-t)(\psi) \le \phi$, which implies $T^*(-t)(\psi) = a\phi$, for some $a \in \mathbb{R}$. Hence, $\psi = aT^*(t)(\phi)$. This proves that $l^1(\Omega)$ is G^*-invariant, and since the operators $T^*(t)$ are lattice isomorphisms, it follows that $DF(BUC^*(\mathbb{R}))$ is also G^*-invariant. $\square$

Theorem 2.8. *$BUC^{\odot}(\mathbb{R}) \subseteq Df(BUC^*(\mathbb{R}))$; that is, $BUC^{\odot}(\mathbb{R})$ is nonatomic.*

Proof. We must show that if $\omega \in \Omega$, then $\delta_\omega \notin BUC^{\odot}(\mathbb{R})$. To illustrate this, we will show that $\forall 0 \neq t \in \mathbb{R}$, $\|T^*(t)\delta_\omega - \delta_\omega\| = 2$.

Since the operators $T^*(t)$, $t \neq 0$, are isometries and $\forall \omega \in \Omega$, $\|\delta_\omega\| = \delta_\omega(e) = 1$, then $\|T^*(t)(\delta_\omega) - \delta_\omega\| \le 2$.

It follows from (4) that

$$\|T^*(t)\delta_\omega - \delta_\omega\| = \|T^*(t)\delta_\omega - \delta_\omega\|_c$$

$$= \sup\left(|(T^*(t)\delta_\omega - \delta_\omega)(f)| : f \in BUC_c(\mathbb{R}) \text{ and } |f| \le e\right).$$

Let $f(x) = exp(iax)$, $x \in \mathbb{R}$, where a is a constant to be determined later. Then for $t \neq 0$,

$$2 \ge |(T^*(t)\delta_\omega - \delta_\omega)(exp(ia\cdot))| = |\delta_\omega\left((T(t) - I)exp(ia\cdot)\right)|$$

$$= |exp(iat) - 1| \cdot |\delta_\omega(exp(ia\cdot))| = |exp(ia) - 1| \cdot |e\hat{x}p(ia\cdot)| = |exp(iat) - 1|.$$

Now, choose a such that at $a \cdot t = \pi$, we obtain

$$2 \ge \|T^*(t)\delta_\omega - \delta_\omega\| \ge 2.$$

Therefore, $\forall t \neq 0$, we have $\|T^*(t)\delta_\omega - \delta_\omega\| = 2$, which implies $\delta_\omega \notin BUC^{\odot}(\mathbb{R})$. $\square$

Remark 2.9. In Section 1, we stated that $C_0^{\odot}(\mathbb{R}) \cong L^1(\mathbb{R})$. Note that this relation illustrates the *nonatomic property* of $C_0^{\odot}(\mathbb{R})$. The nonatomic property of both $C_0^{\odot}(\mathbb{R})$ and $BUC^{\odot}(\mathbb{R})$ follows from the fact that the *additive group* of $\mathbb{R}$ is transitive and has no fixed points. Note that in the above proof, we were able to use a **character** of the *additive group* to illustrate the nonatomic property of $BUC^{\odot}(\mathbb{R})$.

The one-parameter transformation group, given by $\forall t \in \mathbb{R}$, $\Phi(x;t) = x + t$, extends isomorphically to all of Ω. The extension is defined by, $(\forall \omega \in \Omega \setminus \mathbb{R})$ $(\forall t \in \mathbb{R})$,

$$\hat{\Phi}(\omega; t) = \omega'.$$

By Theorem 2.7, $\delta_{\omega'}$ is the unique atom $T^*(t)(\delta_\omega)$. Consequently, the transformation group,

$$\hat{G} = \{\hat{\Phi}(\cdot; t) : t \in \mathbb{R}\},$$

is a group of homeomorphisms of Ω that extends the *additive group* of $\mathbb{R}$ and leaves both $\mathbb{R}$ and $\Omega \setminus \mathbb{R}$ invariant. The action of $\hat{G}$ on $C(\Omega)$ is given by $\forall t \in \mathbb{R}$,

$$\left(T(t)\hat{f}\right)(\omega) = \hat{f}\left(\Phi(\omega; t)\right), \ \omega \in \Omega.$$

Since the *additive group* of $\mathbb{R}$ and $\hat{G}$ are isomorphic, they have the same properties.

3. Translation-invariant, linear functionals on $BUC(\mathbb{R})$

Definition 3.1. A linear functional $\phi \in BUC^*(\mathbb{R})$ is said to be **translation invariant** if $\forall t \in \mathbb{R}$, $T^*(t)(\phi) = \phi$.

We denote FIX as the collection of all *translation invariant* $\phi \in BUC^*(\mathbb{R})$. Observe that

$$FIX \subseteq BUC^{\odot}(\mathbb{R}).$$

In light of Definition 3.1, it seems reasonable to ask whether $FIX \neq 0$, i.e., does $BUC^*(\mathbb{R})$ contain nonzero, translation-invariant, linear functionals? We answer this question in the next theorem. Note that our result is a special case of earlier work done in the theory of amenable groups ([Pie]). However, for the sake of completeness, we include another proof based on the Markov-Kakutani fixed point theorem ([Mar]-[Ka2]).

Theorem 3.2. *$FIX \neq \{0\}$.*

Proof. Let e be the unit in $BUC(\mathbb{R})$. We showed in Section 2 that the set

$$F = \{\phi \in BUC^*(\mathbb{R}) : 0 \leq \phi \text{ and } \|\phi\| = \phi(e) = 1\}$$

is a nonempty, convex, weak*-compact subset of the unit ball in $BUC^*(\mathbb{R})$. Let $G^* = \{T^*(t) : t \in \mathbb{R}\}$ be the adjoint of the translation group G. Then F is invariant under G^*. Moreover, G^* is a commuting, weak*-continuous family of linear operators. Therefore by the [Mar]-[Ka2] fixed point theorems, F has an element which is fixed by each operator in G^*. $\square$

Theorem 3.3. *FIX is a weak*-closed, hence, norm-closed, linear, G^*-invariant vector-sublattice of $BUC^*(\mathbb{R})$. In addition, it is Dedekind complete in $BUC^*(\mathbb{R})$. That is, if $\{\phi_\alpha\}$ is an upward directed system in FIX and $\phi \in BUC^*(\mathbb{R})$, with $0 \leq \phi_\alpha \uparrow \leq \phi$, then $\sup(\phi_\alpha) \in FIX$.*

Proof. From the definition of FIX, it follows that

$$FIX = \bigcap_{t \in \mathbb{R}} ker(I - T^*(t)).$$

For each $t \in \mathbb{R}$, $I - T^*(t)$ is weak*-continuous; it follows that FIX is a weak*-closed, linear subspace of $BUC^*(\mathbb{R})$.

Let $\phi \in FIX$. From Theorem 2.1, we have, $\forall t \in \mathbb{R}$,

$$|\phi| = |T^*(t)(\phi)| = T^*(t)(|\phi|).$$

This shows that $|\phi| \in FIX$; thus, FIX is a vector sublattice of $BUC^*(\mathbb{R})$.

Finally, from the *order-continuity* properties of the operators in G^*, it follows that FIX is *Dedekind complete* in $BUC^*(\mathbb{R})$. $\square$

Recall that in Theorem 2.6 we showed $BUC^*(\mathbb{R}) = M(\mathbb{R}) \oplus C_0^{\perp}(\mathbb{R})$. The next result will show that $BUC^{\odot}(\mathbb{R}) \neq L^1(\mathbb{R})$.

Theorem 3.4. *$FIX \subseteq C_0^{\perp}(\mathbb{R}) = (M(\mathbb{R}))^d$.*

Proof. Because the space of continuous functions with compact support is dense in $C_0(\mathbb{R})$, it suffices to show that ϕ vanishes on $V([a,b])$–the space of continuous functions which vanish outside of a closed, bounded interval $[a,b]$.

Let $0 \le f \in V[a,b]$ and $\phi \ge 0$. Choose $c > (b-a)$ so that the translates $f(\cdot + kc)$, $k = 0, 1, 2, \dots, n$, have disjoint support. Then

$$0 \le \langle \sum_{k=0}^{n} f(\cdot + kc), \phi\rangle = \langle \sum_{k=0}^{n} T(kc)(f), \phi\rangle = \langle f, \sum_{k=0}^{n} T^*(kc)(\phi)\rangle = (n+1)\langle f, \phi\rangle,$$

which implies

$$0 \le (n+1)\phi(f) \le \|\phi\| \sup_{x\in\mathbb{R}} f(x).$$

Dividing the above inequality by $(n+1)$ and letting $n \to \infty$ shows $\phi(f) = 0$. □

We end this section by constructing examples of elements in $(FIX)_1$, where

$$(FIX)_1 = \{\phi \in FIX : \phi(e) = 1\}.$$

Let $x \in \mathbb{R}$ be fixed. $(\forall 0 \ne y \in \mathbb{R})$ $(\forall f \in BUC(\mathbb{R}))$, consider

$$U_y(f(x)) = \frac{1}{y}\int_0^y f(x+s)\ ds.$$

It is obvious that $U_y \in BUC^*(\mathbb{R})$, $U_y \ge 0$, and $\|U_y\| = U_y(e) = 1$. It follows that $U_y \in F$, where F is the set defined in Theorem 3.2. Furthermore, $(FIX)_1 \subseteq F$. We shall show that the weak*-limits of the set $U = \{U_y : y \ne 0\}$ are elements in $(FIX)_1$.

$(\forall f \in BUC(\mathbb{R}))$ $(\forall x, t, y \in \mathbb{R},\ y \ne 0)$, we have

$$|T^*(t)U_y(f(x)) - U_y(f(x))| \le \frac{2|t|\ \|f\|}{|y|}. \tag{5}$$

Therefore, as a function of y,

$$|T^*(t)U_y(f(x)) - U_y(f(x))| = |U_y\,(T(t)f(x) - f(x))\,| \in C_0(\mathbb{R}).$$

Also, $U_y(f(x)) \in C_0(\mathbb{R})$, as a function of y.

In Section 2, we determined that $BUC(\mathbb{R})$ is isometrically lattice-isomorphic to the function space $C(\Omega)$. Thus, for each $\omega \in \Omega \setminus \mathbb{R}$, there exists a net $(y_\tau) \subseteq \mathbb{R}$, such that

$$\hat{f}(\omega) = \lim_{\tau} f(y_\tau).$$

Hence,

$$\lim_{\tau} U_{y_\tau} = \hat{U}_\omega$$

exists as a weak*-limit. Moreover, by (5), the limit exists uniformly in x. Consequently,

$$\lim_{\tau} T^*(t)U_{y_\tau} = \lim_{\tau} U_{y_\tau},$$

which shows $\hat{U}_\omega \in (FIX)_1$. These positive, linear functionals are called **Cesàro means**.

Remark 3.5. Let $WAP(\mathbb{R})$ denote the space of *weakly almost periodic functions* on $\mathbb{R}$ ([Eb]). Then

$$C_0(\mathbb{R}) \subseteq WAP(\mathbb{R}) \subseteq BUC(\mathbb{R}).$$

$WAP(\mathbb{R})$ possesses a unique, translation-invariant mean, defined by

$$M(f) = \lim_{y\to\infty} \left(\frac{1}{y} \int_0^y f(x+s)ds \right),$$

where the convergence is uniform in $x \in \mathbb{R}$. From the uniqueness of M, it follows that if $\phi \in (FIX)_1$, then the restriction of ϕ to $WAP(\mathbb{R})$ coincides with M. In other words, $(FIX)_1$ consists of all the *translation-invariant extensions* of M.

4. Support translation-invariant, linear functionals on $BUC(\mathbb{R})$

Let E be a *Dedekind complete Riesz space* and let $\beta(E)$ be the collection of all *bands* in E. Recall that $\beta(E)$ is a complete *Boolean algebra*. Furthermore, the *bands* generated by a single element in E–the *principal bands*–generate $\beta(E)$. In fact, every *band* B is the supremum of its *principal bands* ([LuZ]).

Definition 4.1. Let $G^* = \{T^*(t) : t \in \mathbb{R}\}$ be the adjoint of the *translation group*. A set $S \subseteq BUC^*(\mathbb{R})$ is said to be **translation invariant** if $\forall t \in \mathbb{R}$, $T^*(t)(S) \subseteq S$.

We shall denote $\beta(G^*)$ as the collection of all *translation invariant* **bands** in $BUC^*(\mathbb{R})$. Observe that $BUC^{\odot}(\mathbb{R}) \in \beta(G^*)$. Thus, to provide a characterization of $BUC^{\odot}(\mathbb{R})$, we must focus on the properties of $\beta(G^*)$.

Theorem 4.2. *$\beta(G^*)$ is a complete, Boolean subalgebra of $\beta(BUC^*(\mathbb{R}))$.*

Proof. The operators in G^* are *lattice isomorphisms*; hence, $\beta(G^*)$ is a subalgebra. To show that $\beta(G^*)$ is complete, assume $(B_\alpha : \alpha \in \{\alpha\})$ is a nonempty subset of $\beta(G^*)$. The *band* $B = \sup(B_\alpha : \alpha \in \{\alpha\})$ can be expressed as $B = (\bigcap_\alpha {B_\alpha}^d)^d$. It follows that $B \in \beta(G^*)$, since the operators in G^* are also *order continuous.* □

Theorem 4.3 (Abstract Radon-Nikodym Theorem)**.** *Suppose E is a Riesz space; let $\tilde{E}$ denote the* **order dual** *of E. Take $0 \le \phi \in \tilde{E}$ and denote $B(\phi)$ as the principal band generated by ϕ. Then $\psi \in B(\phi)$ if and only if $0 \le f_n \downarrow$ in E and $\phi(f_n) \downarrow 0$ implies $|\psi|(f_n) \downarrow 0$ (*[Lux]*).*

If $\psi \in \tilde{E}$, then we define the *support* of ψ as follows:

$$supp(\psi) = (\{f_n \in E : n = 1, 2, \dots\} : 0 \le f_n \downarrow \text{ and } |\psi|(f_n) \downarrow 0).$$

Theorem 4.3 can be restated using this definition of *support*: $\psi \in B(\phi)$ if and only if $supp(\phi) \subseteq supp(\psi)$.

We offer the following characterizations of the *principal bands* in $\beta(G^*)$.

Theorem 4.4. *Let $0 \le \phi \in BUC^*(\mathbb{R})$ and define $Orb(\phi) = \{T^*(t)(\phi) : t \in \mathbb{R}\}$. Then $B(\phi) \in \beta(G^*)$ if and only if $\forall t \in \mathbb{R}$, $T^*(t)(\phi) \in B(\phi)$. In this case, $B(T^*(t)(\phi)) = B(\phi)$, so that every element in $Orb(\phi)$ is a* **weak unit***.*

Proof. It is enough to show that the condition is sufficient. Assume $\psi \in B(\phi)$. Then

$$|\psi| = \sup(\inf(|\psi|, n\phi) : n = 1, 2, \dots).$$

Thus, by the properties of G^*, the following equalities hold:

$$\begin{aligned}\forall t \in \mathbb{R},\ |T^*(t)(\psi)| &= T^*(t)(|\psi|)\\ &= T^*(t)\left(\sup\left(\inf(|\psi|, n\phi) : n = 1, 2, \dots\right)\right)\\ &= \sup\left(\inf(|T^*(t)(\psi)|, nT^*(t)(\phi)) : n = 1, 2, \dots\right).\end{aligned}$$

Hence, $T^*(t)(\psi) \in B(\phi)$, so that $B(\phi) \in \beta(G^*)$. Finally, $B(T^*(t)(\phi)) \subseteq B(\phi)$ and $\phi = T^*(t)(T^*(-t)(\phi))$ imply that $B(\phi) \subseteq B(T^*(t)(\phi))$. Therefore, $\forall t \in \mathbb{R}$, $B(\phi) = B(T^*(t)(\phi))$. □

Theorems 4.3 and 4.4 lead to the following definition.

Definition 4.5. We say that $0 \leq \phi \in BUC^*(\mathbb{R})$ is **support translation-invariant** if $\forall t \in \mathbb{R}$, $T^*(t)(\phi) \in B(\phi)$; that is, if $B(\phi) \in \beta(G^*)$.

We designate $SFIX$ as the collection of all *support translation-invariant*, linear functionals in $BUC^*(\mathbb{R})$. Clearly, $\phi \in SFIX$ if and only if $\forall t \in \mathbb{R}$, $supp(T^*(t)(\phi)) = supp(\phi)$. Moreover, if $(FIX)_+ = \{\phi \in FIX : \phi \geq 0\}$, then $(FIX)_+ \subseteq SFIX$.

Let $\beta_p(G^*)$ be the collection of all *translation-invariant*, **principal bands** in $BUC^*(\mathbb{R})$; let $(BUC^*(\mathbb{R}))_+$ represent the positive, linear functionals in $BUC^*(\mathbb{R})$. Then

Theorem 4.6. *SFIX is translation-invariant, Dedekind complete, and norm-closed. In addition, $SFIX$ is a sublattice-ordered cone in $(BUC^*(\mathbb{R}))_+$. Also, $\beta_p(G^*)$ is a* **complete ideal** *in $\beta(G^*)$.*

Proof. If $\phi_1, \phi_2 \in SFIX$, then $B(\phi_1), B(\phi_2) \subseteq B(\phi_1 + \phi_2)$; thus, $\forall t \in \mathbb{R}$,

$$T^*(t)(\phi_1) + T^*(t)(\phi_2) \in B(\phi_1) \oplus B(\phi_2) \subseteq B(\phi_1 + \phi_2).$$

If $\lambda \geq 0$, then $B(\lambda\phi) = B(\phi)$ shows that $SFIX$ is a *cone* in $(BUC^*(\mathbb{R}))_+$.

It follows from $B(\inf(\phi_1, \phi_2)) = B(\phi_1) \cap B(\phi_2)$ and $B(\sup(\phi_1, \phi_2)) = B(\phi_1 + \phi_2)$ that $SFIX$ is a sublattice.

A trivial check shows $SFIX$ is translation invariant. To show that $SFIX$ is *Dedekind complete*, let $\phi_\alpha \in SFIX$ with $\phi_\alpha \uparrow \phi$ in $BUC^*(\mathbb{R})$. Then $\sup_\alpha B(\phi_\alpha) = B(\phi)$. Hence, $\forall t \in \mathbb{R}$,

$$T^*(t)(\sup B(\phi_\alpha)) = \sup B(T^*(t)(\phi_\alpha)) = \sup B(\phi_\alpha).$$

Let $\phi_n \in SFIX$ with $\|\phi_n - \phi\| \to 0$ in $BUC^*(\mathbb{R})$. Since $BUC^*(\mathbb{R})$ has *order continuous norm* (see Section 2), there exists a subsequence ϕ_{nk} such that $\inf(\phi_{nk+l} : l \geq 0) \uparrow \phi$. Therefore, $\phi \in SFIX$.

That $\beta_p(G^*)$ is an ideal in $\beta(G^*)$ follows from the fact that $BUC^*(\mathbb{R})$ is *Dedekind complete*. Recall from Theorem 3 that every *band* in a *Dedekind complete* space is a *projection band*. Ergo, if B is a *band* and $B \subseteq B(\phi)$, then B is generated by the *projection* of ϕ on B. □

From the classical *Radon-Nikodym theorem*, $L^1(\mathbb{R})$ is a *band* in $M(\mathbb{R})$, which implies that it is also a *band* in $BUC^*(\mathbb{R})$.

Theorem 4.7. *$M(\mathbb{R}) \in \beta(G^*)$ and $L^1(\mathbb{R}) \in \beta_p(G^*)$. Also, $L^1(\mathbb{R})$ is an* **atom** *of $\beta(G^*)$, i.e., if $B \in \beta(G^*)$ and $B \subseteq L^1(\mathbb{R})$, then $B = \{0\}$ or $B = L^1(\mathbb{R})$.*

Proof. Clearly, $M(\mathbb{R}) \in \beta(G^*)$. To show $L^1(\mathbb{R}) \in \beta_p(G^*)$, we first show $L^1(\mathbb{R})$ is a *principal band*. But this follows from the fact that $L^1(\mathbb{R})$ contains functions $f(x) > 0$, for all $x \in \mathbb{R}$. Thus, if $0 \le gdx \in L^1(\mathbb{R})$, then we can write

$$gdx = \sup(\inf(g, nf)dx : n = 1, 2, \dots);$$

this shows that $L^1(\mathbb{R})$ is a *principal band*.

That $L^1(\mathbb{R})$ is a translation-invariant, principal band follows from the translation invariance of Lebesgue measure. That is, $\forall t \in \mathbb{R}$,

$$T^*(t)(g(\cdot)dx) = g(\cdot - t)dx \in L^1(\mathbb{R}).$$

Therefore, $L^1(\mathbb{R}) \in \beta_p(G^*)$. □

To prove $L^1(\mathbb{R})$ is an *atom*, assume that $B \in \beta(G^*)$ satisfies $B \subseteq L^1(\mathbb{R})$. Since $L^1(\mathbb{R})$ is a *principal band*, then B is also. Hence, $B = B(fdx)$, for some $0 \le f \in L^1(\mathbb{R})$. Let $F = \{x : f(x) > 0\}$. Since B is translation invariant, then

$$T^*(t)(fdx) = f(\cdot - t)dx \in B.$$

Thus, the *support* of F is translation invariant. From the property that the *Lebesgue-measure* is *ergodic* with respect to the *translation group*, we conclude that either F is a set of measure zero or $F = \mathbb{R}$, a.e. $[dx]$.

The next result will show that $L^1(\mathbb{R})$ is the only *translation-invariant, prinicipal band* in $M(\mathbb{R})$.

Theorem 4.8. *If $\phi \in SFIX \cap M(\mathbb{R})$ and $\phi \neq 0$, then $B(\phi) = L^1(\mathbb{R})$.*

Proof. Assume that $0 \le fdx$ is a *weak unit* in $L^1(\mathbb{R})$ and that $0 \le g_n \in BUC(\mathbb{R})$, $n = 1, 2, \dots$, satisfy $0 \le g_n \downarrow 0$ a.e. $[dx]$. First, observe that $\forall n \ge 1$,

$$h_n(t) = \int_{-\infty}^{+\infty} g_n(x+t)f(x)dx \in BUC(\mathbb{R}).$$

Also, $\forall t \in \mathbb{R}$, $h_n(t) \downarrow 0$. From $\phi \in M(\mathbb{R})$, it follows that $\phi(h_n) \downarrow 0$. From *Fubini's theorem*,

$$\phi(h_n) = \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} g_n(t)d(T^*(x)\phi)(t) \right) f(x)dx, \ n = 1, 2, \dots.$$

Hence, for almost all $x \in \mathbb{R}$, $\langle g_n, T^*(x)(\phi) \rangle \downarrow 0$, which implies that $T^*(x)(\phi) \in L^1(\mathbb{R})$, for these x-values. It follows from Theorem 4.4 that since $\phi \in SFIX$, then $\forall x \in \mathbb{R}$, $B(T^*(x)(\phi)) \subseteq L^1(\mathbb{R})$. Therefore, by Theorem 4.7 and the fact that $\phi \neq 0$, $B(\phi) = L^1(\mathbb{R})$. □

Corollary 4.9. *The* **disjoint complement** *of $L^1(\mathbb{R})$ in $M(\mathbb{R})$ has no weak unit.*

Proof. Suppose $\phi > 0$ is a *weak unit* in $D = (L^1(\mathbb{R}))^d \cap M(\mathbb{R})$. Then $D = B(\phi)$. Since $D \in \beta(G^*)$, then $\forall t \in \mathbb{R}$, $T^*(t)(\phi) \in D$. But this implies that $\phi \in SFIX \cap M(\mathbb{R})$, which is a contradiction of Theorem 4.8. □

5. Characterizations of $BUC^{\odot}(\mathbb{R})$

Recall

$$BUC^*(\mathbb{R}) = BUC^{\odot}(\mathbb{R}) \oplus (BUC^{\odot}(\mathbb{R}))^d.$$

The above equation shows that any $\phi \in BUC^*(\mathbb{R})$ can be written as $\phi = \phi^{\odot} + \phi^d$, where $\phi^{\odot}$ [resp. ϕ^d] is the part of ϕ in $BUC^{\odot}(\mathbb{R})$ [resp. $(BUC^{\odot}(\mathbb{R}))^d$]. These facts lead to

Lemma 5.1. *If $\phi \in SFIX$, then $\phi^{\odot}, \phi^d \in SFIX$.*

Proof. The bands in $BUC^*(\mathbb{R})$ form a Boolean algebra; hence,

$$B(\phi) = B(\phi^{\odot}) \oplus B(\phi^d).$$

But $\phi \in SFIX$, so that $\forall t \in \mathbb{R}$, $T^*(t)(\phi^{\odot}) + T^*(t)(\phi^d) = T^*(t)(\phi)$ implies

$$B(T^*(t)(\phi^{\odot})) \oplus B(T^*(t)(\phi^d)) = B(T^*(t)(\phi)) = B(\phi).$$

Since $BUC^{\odot}(\mathbb{R}), (BUC^{\odot}(\mathbb{R}))^d \in \beta(G^*)$, then $T^*(t)(\phi^{\odot})$ and $T^*(t)(\phi^d)$ are in $BUC^{\odot}(\mathbb{R})$ and $(BUC^{\odot}(\mathbb{R}))^d$, respectively. Therefore,

$$B(\phi^{\odot}) \oplus B(\phi^d) = B(\phi) = B(T^*(t)(\phi^{\odot})) \oplus B(T^*(t)(\phi^d))$$

shows that $\forall t \in \mathbb{R}$,

$$B(T^*(t)(\phi^{\odot})) = B(\phi^{\odot}) \text{ and } B(T^*(t)(\phi^d)) = B(\phi^d).$$

Consequently, $\phi^{\odot}, \phi^d \in SFIX$ (Theorem 4.4). $\square$

We are now in a position to prove the first of our main theorems.

Theorem 5.2. *$SFIX \subseteq BUC^{\odot}(\mathbb{R})$.*

Proof. Let $\phi \in SFIX$. Then by (Lemma 5.1),

$$\phi = \phi^{\odot} + \phi^d, \text{ with } (\phi^{\odot}, \phi^d) \in SFIX.$$

Let A^* be the *weak**-generator of G^*, let $\rho(A^*)$ be the resolvent set of A^*, and let $R(\lambda, A^*)$, $0 < \lambda \in \rho(A^*)$, be the resolvent of A^*. It is well known that $R(\lambda, A^*)$ satisfies the following *weak**-integral: $(\forall \phi \in BUC^*(\mathbb{R}))$ $(\forall f \in BUC(\mathbb{R}))$,

$$\langle f, R(\lambda, A^*)(\phi)\rangle = \langle f, \int_0^{\infty} e^{-\lambda t} T^*(t)(\phi)\, dt\rangle = \int_0^{\infty} e^{-\lambda t}\langle f, T^*(t)(\phi)\rangle\, dt. \tag{6}$$

We will show, using the *weak**-integral equations above, that

$$R(\lambda, A^*)(\phi^d) \in B(\phi^d) \subseteq (BUC^{\odot}(\mathbb{R}))^d.$$

Since we also have

$$R(\lambda, A^*)(\phi^d) \in dom(A^*) \subseteq BUC^{\odot}(\mathbb{R}),$$

where $dom(A^*)$ is the domain of A^*, then $R(\lambda, A^*)(\phi^d) = 0$. But $R(\lambda, A^*)$ is invertible; hence, $\phi^d = 0$, so that $\phi = \phi^d + \phi^{\odot} = \phi^{\odot} \in BUC^{\odot}(\mathbb{R})$.

The fact that $\phi^d \in SFIX$ implies, $\forall t \in \mathbb{R}$, $T^*(t)(\phi^d) \in B(\phi^d)$. Let $0 \le f_n \in BUC(\mathbb{R})$, with $f_n \downarrow$ and $\phi^d(f_n) \downarrow 0$. Then

$$\left(T^*(t)(\phi^d)\right)(f_n) \downarrow 0.$$

But this means

$$\langle f_n, R(\lambda, A^*)(\phi^d)\rangle \downarrow 0, \text{by } (6).$$

Hence,

$$R(\lambda, A^*)(\phi^d) \in B(\phi^d) \subseteq (BUC^{\odot}(\mathbb{R}))^d \text{ (Theorem 4.3).}$$

$\square$

Corollary 5.3. *If $\phi \in (BUC^{\odot}(\mathbb{R}))^d$ and $B(\phi) \in \beta_p(G^*)$, then $\phi = 0$.*

Let $Orb(\phi) = \{T^*(t)(\phi) : t \in \mathbb{R}\}$; we will use this set to prove the last of our main results (Theorem 5.5).

Theorem 5.4. *If $0 \leq \phi \in BUC^*(\mathbb{R})$ and $Orb(\phi)$ is norm-separable, then there exists $\chi \in SFIX$ such that $\phi \leq \chi$. Consequently, $BUC^{\odot}(\mathbb{R})$ is the* **order ideal** *in $BUC^*(\mathbb{R})$ that is generated by $SFIX$. In other words, $SFIX$ is a majorizing subcone of $BUC^{\odot}(\mathbb{R})$.*

Proof. From the hypothesis, it follows that there exists a countable subset $\{r_n\}_{n=1}^{\infty}$ of $\mathbb{R}$ such that $\{T^*(r_n)(\phi)\}_{n=1}^{\infty}$ is norm-dense in $Orb(\phi)$. Let $\varepsilon_n > 0$, $n = 1, 2, \ldots,$ and assume $\sum_{n=1}^{\infty} \varepsilon_n \|T^*(r_n)(\phi)\| < \infty$. Define

$$\chi = \phi + \sum_{n=1}^{\infty} \varepsilon_n T^*(r_n)(\phi). \tag{7}$$

We will show $\chi \in SFIX$.

Consider $B(\chi)$, the principal band generated by χ. We claim that, $\forall t \in \mathbb{R}$, $T^*(t)(\phi) \in B(\chi)$. First, observe $\phi \leq \chi$, and so $\phi \in B(\chi)$, since $B(\chi)$ is an *order ideal*. Next, from the definiton of χ, $\forall n \geq 1$,

$$\varepsilon_n T^*(r_n)(\phi) \leq \chi \Rightarrow T^*(r_n)(\phi) \in B(\chi).$$

Let $t \in \mathbb{R}$. Then there exists a subsequence $\{T^*(r_{n(k)})(\phi)\}$ that converges to $T^*(t)(\phi)$, in the norm. But this implies $T^*(t)(\phi) \in B(\chi)$, since $B(\chi)$ is norm-closed. Moreover, by (7), $\forall t \in \mathbb{R}$,

$$T^*(t)(\chi) = T^*(t)(\phi) + \sum_{n=1}^{\infty} \varepsilon_n T^*(t)T^*(r_n)(\phi) = T^*(t)(\phi) + \sum_{n=1}^{\infty} \varepsilon_n T^*(t + r_n)(\phi).$$

Clearly, the last term in the above equality is contained in $B(\chi)$; hence, $\chi \in SFIX$.

Let $\phi \in BUC^{\odot}(\mathbb{R})$. Since the operators in G^* are strongly continuous on $BUC^{\odot}(\mathbb{R})$, it follows that $Orb(\phi)$ is norm-separable. Hence, there exists $\chi \in SFIX$, with $|\phi| \leq \chi$. This shows that the order ideal or solid hull of $SFIX$ is $BUC^{\odot}(\mathbb{R})$. □

The following theorem is a consequence of Theorem 5.4.

Theorem 5.5. *$\phi \in BUC^{\odot}(\mathbb{R})$ if and only if $Orb(\phi)$ is norm-separable. In particular, the translation-invariant bands, $(BUC^{\odot}(\mathbb{R}))^d \cap M(\mathbb{R})$ and $(BUC^{\odot}(\mathbb{R}))^d \cap C_0^{\perp}(\mathbb{R})$, are not norm-separable.*

We conclude our discussion by applying the following *Wiener-Young* type theorem to $BUC(\mathbb{R})$ ([Jvn]).

Theorem 5.6. *If $\phi \in BUC^*(\mathbb{R})$, then*

$$\lim_{t \to 0} \sup(\|T^*(t)(\phi) - \phi\|) = 2\|\phi^d\|.$$

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